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The Best Asymptotic Constant of a Class of Approximation Operators*

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The best asymptotic constant was established by Esseen for Bernstein operators. In this paper, we extend Esseen's result to a class of linear positive operators and as byproduct we obtain the best asymptotic constant for Szász, Baskakov, Gamma, and B-spline operators.

1. INTRODUCTION

If f is a function defined on $[0, 1]$, the Bernstein polynomial order n of the function $f(x)$ is defined by

$$B_n(f, x) = \sum_{k=0}^{n} f\left(\frac{k}{n}\right) P_{n,k}(x), \tag{1.1}$$

where

$$P_{n,k}(x) = \binom{n}{k} x^k (1-x)^{n-k}.$$

Let

$$\mu = \sup_n \sup_f \max_{0 \leqslant x \leqslant 1} \frac{|B_n(f, x) - f(x)|}{\omega(f, n^{-1/2})}, \tag{1.2}$$

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where $\omega(f, h)$ is the modulus of continuity of f on $[0, 1]$, and the second supremum is taken over all $f \in C[0, 1]$. Sikkema [7] has shown that

$$\mu = \frac{4306 + 837\sqrt{6}}{5832} = 1.0898873....$$

μ is the best constant in the sense of being the smallest μ' for which

$$\max_{0 \leqslant x \leqslant 1} |B_n(f, x) - f(x)| \leqslant \mu' \omega(f, n^{-1/2}) \tag{1.3}$$

holds for every $f \in C[0, 1]$, and $n = 1, 2, 3, ...$ Besides the best constant, people are also interested in the best asymptotic constant. Let

$$\gamma_n = \sup_f \max_{0 \leqslant x \leqslant 1} \frac{|B_n(f, x) - f(x)|}{\omega(f, n^{-1/2})} \tag{1.4}$$

and $\gamma = \limsup_{n \to \infty} \gamma_n$. Then γ is called the best asymptotic constant. From the definition of γ, for every $\varepsilon > 0$, there exists an integer n_0 such that, if $n \geqslant n_0$,

$$\max_{0 \leqslant x \leqslant 1} |B_n(f, x) - f(x)| \leqslant (\gamma + \varepsilon)\, \omega(f, n^{-1/2}) \tag{1.5}$$

for every $f \in C[0, 1]$, and for some $f \in C[0, 1]$ and $n \geqslant n_0$,

$$\max_{0 \leqslant x \leqslant 1} |B_n(f, x) - f(x)| \geqslant (\gamma - \varepsilon)\, \omega(f, n^{-1/2}). \tag{1.6}$$

Esseen [5] has shown that

$$\gamma = 2 \sum_{i=0}^{\infty} (i+1)(\Phi(2i+2) - \Phi(2i)) = 1.045564, \tag{1.7}$$

where

$$\Phi(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} e^{-t^2/2}\, dt.$$

An alternative consideration is to find the best asymptotic constant for each fixed x. Let $\{L_n\}$ be a sequence of positive linear operators on $C[a, b]$. With appropriate conditions on $T_{n,j}(x) = L_n((u-x)^j, x)$, $j = 0, 1, 2, ...,$ Zhou [8, 9] obtained the best asymptotic constant for each fixed x with $f(x) \in W^{r+\alpha}M$, where $W^{r+\alpha}M = \{f : f^{(r)} \in \mathrm{Lip}_M(\alpha)\}$. Zhou's results require that $T_{n,j}(x)$ be finite, which may not be satisfied for some positive linear operators. We are more interested in the asymptotic constant with uniform properties as described by (1.5) and (1.6).

In this note, we study the best asymptotic constant in Esseen's sense for a class of operators which have the form

$$L_n(f, x) = E_x f(T_n) = \int_\Omega f(T_n)\, dP_x = \int_{-\infty}^{\infty} f\left(\frac{t}{n}\right) dF_{n,x}(t), \tag{1.8}$$

where $T_n = (1/n) \sum_{i=1}^n Y_i$, $Y_1, ..., Y_n$ are iid (independent and identically distributed) random variables on a probability space $(\Omega, \mathscr{F}, P_x)$ with expectation $E_x Y_1 = x$, x is a parameter taking values in an interval I, $F_{n,x}(t)$ is the distribution function of $\sum_{i=1}^n Y_i$, and $f(x)$ is continuous on the real line $R = (-\infty, \infty)$. Bernstein, Szász, Baskakov, Gamma, and Weierstrass operators are the special cases of this class of operators (see Khan [6]). The main result are given in Section 2. We shall see that, under mild conditions, the best asymptotic constant exists and is easy to calculate. In Section 3, we find the best asymptotic constant for some well-known operators. The main tools used in this paper are the Chebyshev inequality, Lebesgue's dominated convergence theorem, and the central limit theorem, which can be found in most text books in probability, such as Billingley [1] or Chung [3].

2. Main Results

Consider operators defined by (1.8).

THEOREM 1. *If*

$$\sup_{x \in I} n^{1/2} E_x |T_n - x| = O(1), \tag{2.1}$$

then

$$\gamma_n = \sup_{x \in I} \sum_{k=0}^{\infty} P_x(n^{1/2} |T_n - x| \geqslant k) = O(1). \tag{2.2}$$

Proof. By a straightforward calculation, we have

$$\sum_{k=0}^{\infty} P_x(n^{1/2} |T_n - x| \geqslant k) \leqslant 1 + n^{1/2} E_x |T_n - x|.$$

Hence, by (2.1),

$$\sup_{x \in I} \sum_{k=0}^{\infty} P_x(n^{1/2} |T_n - x| \geqslant k) = O(1).$$

Let

$$A_k = \{k \leqslant n^{1/2} |T_n - x| < k+1\}, \qquad k = 0, 1, 2, \ldots .$$

Since

$$\begin{aligned} |E_x f(T_n) - f(x)| &\leqslant E_x |f(T_n) - f(x)| \\ &\leqslant \sum_{k=0}^{\infty} \int_{A_k} |f(T_n) - f(x)| \, dP_x \\ &\leqslant \omega(f, n^{-1/2}) \sum_{k=0}^{\infty} P_x(n^{1/2} |T_n - x| \geqslant k), \end{aligned} \tag{2.3}$$

we have

$$\sup_{x \in I} \sup_{f} \frac{|E_x f(T_n) - f(x)|}{\omega(f, n^{-1/2})} \leqslant \sup_{x \in I} \sum_{k=0}^{\infty} P_x(n^{1/2} |T_n - x| \geqslant k). \tag{2.4}$$

For each fixed $x_0 \in I$, consider the function $g_{n,\varepsilon}(x)$ on $[x_0, \infty)$,

$$g_{n,\varepsilon}(x) = \begin{cases} k + \frac{1}{\varepsilon}(x - x_0 - kn^{-1/2}) & \\ \qquad\qquad x_0 + kn^{-1/2} \leqslant x \leqslant x_0 + kn^{-1/2} + \varepsilon & \\ k+1 \qquad x_0 + kn^{-1/2} + \varepsilon \leqslant x \leqslant x_0 + (k+1)n^{-1/2}, & \end{cases}$$

where $k = 0, 1, 2, \ldots, 0 < \varepsilon < n^{-1/2}$. Also define $f_{n,\varepsilon}(x)$ on $(-\infty, \infty)$,

$$f_{n,\varepsilon}(x) = \begin{cases} g_{n,\varepsilon}(x) & \text{if} \quad x \geqslant x_0 \\ g_{n,\varepsilon}(2x_0 - x) & x \leqslant x_0. \end{cases}$$

For each $n, f_{n,\varepsilon}(x)$ is continuous, symmetric about $x = x_0$, and $\omega(f_{n,\varepsilon}, n^{-1/2}) = 1$, $f_{n,\varepsilon}(x_0) = 0$. Since $\sum_{k=0}^{\infty} (k+1) P_{x_0}(A_k) < \infty$, Lebesgue's dominated convergence theorem implies that

$$\begin{aligned} \lim_{\varepsilon \to 0} \frac{|E_{x_0} f_{n,\varepsilon}(T_n) - f_{n,\varepsilon}(x_0)|}{\omega(f_{n,\varepsilon}, n^{-1/2})} &= \lim_{\varepsilon \to 0} E_{x_0} f_{n,\varepsilon}(T_n) \\ &= \lim_{\varepsilon \to 0} \sum_{k=0}^{\infty} \int_{A_k} f_{n,\varepsilon}(T_n) \, dP_{x_0} \\ &= \sum_{k=0}^{\infty} \int_{A_k} (k+1) \, dP_{x_0} \\ &= \sum_{k=0}^{\infty} P_{x_0}(n^{1/2} |T_n - x_0| \geqslant k). \end{aligned} \tag{2.5}$$

Hence

$$\sup_{x\in I}\sup_{f}\frac{|E_x f(T_n)-f(x)|}{\omega(f, n^{-1/2})}\geqslant \sup_{x\in I}\sum_{k=0}^{\infty} P_x(n^{1/2}\,|T_n-x|\geqslant k). \tag{2.6}$$

Combining (2.4) and (2.6), we have

$$\sup_{x\in I}\sup_{f}\frac{|E_x f(T_n)-f(x)|}{\omega(f, n^{-1/2})}= \sup_{x\in I}\sum_{k=0}^{\infty} P_x(n^{1/2}\,|T_n-x|\geqslant k). \tag{2.7}$$

To complete the proof, one needs to show that $\sup_{x\in I}$ and $\sup_f$ are interchangeable on the left hand side of (2.7). First, by (2.3),

$$\sup_{x\in I}\frac{|E_x f(T_n)-f(x)|}{\omega(f, n^{-1/2})}\leqslant \sup_{x\in I}\sum_{k=0}^{\infty} P_x(n^{1/2}\,|T_n-x|\geqslant k).$$

Thus, by (2.7),

$$\sup_{f}\sup_{x\in I}\frac{|E_x f(T_n)-f(x)|}{\omega(f, n^{-1/2})}\leqslant \sup_{x\in I}\sup_{f}\frac{|E_x f(T_n)-f(x)|}{\omega(f, n^{-1/2})}. \tag{2.8}$$

On the other hand, for each fixed $\tilde{x}\in I$, there exists a sequence $\{f_m\}$,

$$\sup_{f}\frac{|E_{\tilde{x}} f(T_n)-f(\tilde{x})|}{\omega(f, n^{-1/2})}= \lim_{m\to\infty}\frac{|E_{\tilde{x}} f_m(T_n)-f_m(\tilde{x})|}{\omega(f_m, n^{-1/2})}, \tag{2.9}$$

and for $m=1, 2, 3, \ldots$,

$$\begin{aligned}\frac{|E_{\tilde{x}} f_m(T_n)-f_m(\tilde{x})|}{\omega(f_m, n^{-1/2})}&\leqslant \sup_{x\in I}\frac{|E_x f_m(T_n)-f_m(x)|}{\omega(f_m, n^{-1/2})}\\ &\leqslant \sup_{f}\sup_{x\in I}\frac{|E_x f(T_n)-f(x)|}{\omega(f, n^{-1/2})}.\end{aligned} \tag{2.10}$$

Hence, by (2.9) and (2.10),

$$\sup_{x\in I}\sup_{f}\frac{|E_x f(T_n)-f(x)|}{\omega(f, n^{-1/2})}\leqslant \sup_{f}\sup_{x\in I}\frac{|E_x f(T_n)-f(x)|}{\omega(f, n^{-1/2})}. \tag{2.11}$$

When (2.8) and (2.11) are combined, the proof is complete.

Remark. In Theorem 1, we do not require that the operators have the form (1.8). In some special cases, we can find γ_n for each n by using Theorem 1.

EXAMPLE. Let $Y_1, ..., Y_n$ be iid random variables with density function $g(t, x) = (1/\sqrt{2\pi})\, e^{-(t-x)^2/2}$. Then T_n has density function $g_n(t, x) = (n/2\pi)^{1/2}\, e^{-n(t-x)^2/2}$. The Weierstrass operator is

$$W_n(f, x) = E_x f(T_n) = \left(\frac{n}{2\pi}\right)^{1/2} \int_{-\infty}^{\infty} f(t)\, e^{-n(t-x)^2/2}\, dt,$$

where $x \in I$, $E_x\, |f(T_n)| < \infty$. Since

$$n^{1/2} E_x\, |T_n - x| \leqslant [nE_x(T_n - x)^2]^{1/2} = 1,$$

the condition of Theorem 1 holds. Since $n^{1/2}(T_n - x) = Z$ has a standard normal distribution, by Theorem 1,

$$\begin{aligned}\gamma_n &= \sup_{x \in I} \sum_{k=0}^{\infty} P_x(n^{1/2}\, |T_n - x| \geqslant k) \\ &= \sum_{k=0}^{\infty} P(|Z| \geqslant k) = 2 \sum_{k=0}^{\infty} (1 - \Phi(k)) = 1.365574.\end{aligned}$$

The main result of this section is stated as follows.

THEOREM 2. *Suppose that* $v(x) = E_x(Y_1 - x)^2$ *is continuous and has finitely many zeros on a finite interval* I. *Assume furthermore that* $w(x) = E_x\, |Y_1 - x|^3$ *is bounded on* I. *Then*

$$\gamma = \limsup_{n \to \infty} \gamma_n = 2 \sum_{k=0}^{\infty} \left(1 - \Phi\left(\frac{k}{\sqrt{M}}\right)\right), \tag{2.12}$$

where $M = \max_{x \in I} v(x)$.

Proof. Since

$$n^{1/2} E_x\, |T_n - x| \leqslant [nE_x(T_n - x)^2]^{1/2} = v(x)^{1/2},$$

the condition of Theorem 1 holds. Thus,

$$\gamma_n = \sup_{x \in I} \sum_{k=0}^{\infty} P_x(n^{1/2}\, |T_n - x| \geqslant k). \tag{2.13}$$

To prove (2.12), it suffices to show that

$$\limsup_{n \to \infty} \sup_{x \in I} \sum_{k=0}^{\infty} P_x(n^{1/2}\, |T_n - x| \geqslant k) = 2 \sum_{k=0}^{\infty} \left(1 - \Phi\left(\frac{k}{\sqrt{M}}\right)\right). \tag{2.14}$$

Let $\{\gamma_{n_m}\}$ be a subsequence of $\{\gamma_n\}$, $\gamma = \lim_{m\to\infty} \gamma_{n_m}$, there exists $x_m \in I$,

$$\gamma_{n_m} - \frac{1}{m} < \sum_{k=0}^{\infty} P_{x_m}(n_m^{1/2} \,|T_{n_m} - x_m| \geq k) \leq \gamma_{n_m}. \tag{2.15}$$

Since I is compact, there exist $x_0 \in I$ and a subsequence $\{x_{m_i}\}$ of $\{x_m\}$, $v(x_{m_i}) \neq 0$, $\lim_{i\to\infty} x_{m_i} = x_0$, such that

$$\gamma_{n_{m_i}} - \frac{1}{m_i} < \sum_{k=0}^{\infty} P_{x_{m_i}}(n_{m_i}^{1/2} \,|T_{n_{m_i}} - x_{m_i}| \geq k) \leq \gamma_{n_{m_i}}. \tag{2.16}$$

We first show that $v(x_0) \neq 0$. From the conditions of Theorem 2, there exists $\hat{x} \in I$, $v(\hat{x}) = \max_{x\in I} v(x) > 0$. By Lebesgue's dominated convergence theorem and the central limit theorem,

$$\lim_{n\to\infty} \sum_{k=0}^{\infty} P_{\hat{x}}(n^{1/2} \,|T_n - \hat{x}| \geq k) = 1 + \Delta(\hat{x}), \tag{2.17}$$

where

$$\Delta(\hat{x}) = \sum_{k=0}^{\infty} P\left(|Z| \geq \frac{k}{\sqrt{v(\hat{x})}}\right) = 2 \sum_{k=0}^{\infty} \left(1 - \Phi\left(\frac{k}{\sqrt{v(\hat{x})}}\right)\right) > 0.$$

Hence

$$\gamma = \limsup_{n\to\infty} \gamma_n \geq 1 + \Delta(\hat{x}) > 1. \tag{2.18}$$

If $v(x_0) = 0$, by the Chebyshev inequality,

$$\sum_{k=0}^{\infty} P_{x_{m_i}}(n_{m_i}^{1/2} \,|T_{n_{m_i}} - x_{m_i}| \geq k) \leq 1 + v(x_{m_i}) \sum_{k=0}^{\infty} \frac{1}{k^2}, \tag{2.19}$$

implying that

$$\lim_{i\to\infty} \sum_{k=0}^{\infty} P_{x_{m_i}}(n_{m_i}^{1/2} \,|T_{n_{m_i}} - x_{m_i}| \geq k) \leq 1. \tag{2.20}$$

Thus $\gamma \leq 1$ by (2.16), which is in contradiction with (2.18).

Let $\bar{x}_1, ..., \bar{x}_s$ be the zeros of $v(x)$ on I. From the above proof, there is a $\delta > 0$ such that

$$x_{m_i} \notin A_s = I \cap \left(\bigcup_{i=1}^{s} (\bar{x}_i - \delta, \bar{x}_i + \delta)\right).$$

Since $A = I - A_s$ is compact, we have $b = \min_{x \in A} v(x)^{1/2} > 0$. By the Berry–Esseen theorem [2],

$$\sup_{x_{m_i} \in A} \sup_{k \geqslant 0} \left| P_{x_{m_i}}(n_{m_i}^{1/2} \, |T_{n_{m_i}} - x_{m_i}| \geqslant k) - P\left(|Z| \geqslant \frac{k}{\sqrt{v(x_{m_i})}} \right) \right| \leqslant \frac{2 \sup_{x \in I} w(x)}{b^3 n_{m_i}^{1/2}}. \tag{2.21}$$

Thus

$$\lim_{i \to \infty} P_{x_{m_i}}(n_{m_i}^{1/2} \, |T_{n_{m_i}} - x_{m_i}| \geqslant k) = P\left(|Z| \geqslant \frac{k}{\sqrt{v(x_0)}} \right), \tag{2.22}$$

for $k = 1, 2, \ldots$, implying, by Lebesgue's dominated convergence theorem,

$$\lim_{i \to \infty} \sum_{k=0}^{\infty} P_{x_{m_i}}(n_{m_i}^{1/2} \, |T_{n_{m_i}} - x_{m_i}| \geqslant k) = \sum_{k=0}^{\infty} P\left(|Z| \geqslant \frac{k}{\sqrt{v(x_0)}} \right).$$

Therefore, by (2.16),

$$\gamma = \sum_{k=0}^{\infty} P\left(|Z| \geqslant \frac{k}{\sqrt{v(x_0)}} \right).$$

Finally, in order to show that $v(x_0) = M = \max_{x \in I} v(x)$, let $\bar{x} \in I$, $M = v(\bar{x})$. Since

$$\gamma_n \geqslant \sum_{k=0}^{\infty} P_{\bar{x}}(n^{1/2} \, |T_n - \bar{x}| \geqslant k),$$

$$\gamma \geqslant \sum_{k=0}^{\infty} P\left(|Z| \geqslant \frac{k}{\sqrt{v(\bar{x})}} \right).$$

Thus

$$\sum_{k=0}^{\infty} P\left(|Z| \geqslant \frac{k}{\sqrt{v(\bar{x})}} \right) \leqslant \sum_{k=0}^{\infty} P\left(|Z| \geqslant \frac{k}{\sqrt{v(x_0)}} \right),$$

which implies that $v(x_0) \geqslant v(\bar{x}) = M$. The proof is complete.

3. Examples

(1) Bernstein Operator. Let $P_x(Y_1 = 1) = 1 - P_x(Y_1 = 0) = x$, $x \in [0, 1]$. Then (1.8) defines the Bernstein polynomials given by (1.1).

With a simple calculation, $v(x)=x(1-x)$, $\max_{x\in I} v(x)=\frac{1}{4}$, and $w(x)$ is bounded on [0, 1]. By Theorem 2,

$$\gamma=2\sum_{k=0}^{\infty}(1-\Phi(2k))=1.045564. \tag{3.1}$$

This is the result of Esseen [5].

(2) SZÁS OPERATOR. If $P_x(Y_1=k)=x^k e^{-x}/k!$, $k=0, 1, 2, ..., x\in[0, a]$, $0<a<\infty$, then (1.8) defines the Szász operator

$$S_n(f, x)=e^{-nx}\sum_{k=0}^{\infty} f\left(\frac{k}{n}\right)\frac{(nx)^k}{k!}, \tag{3.2}$$

and $v(x)=x$, $\max_{x\in I} v(x)=a$. Following Theorem 2,

$$\gamma=2\sum_{k=0}^{\infty}\left(1-\Phi\left(\frac{k}{\sqrt{a}}\right)\right). \tag{3.3}$$

(3) GAMMA OPERATOR. Let the density function of Y_1 be $g(t, x)=x^{-1} e^{-t/x}$, $t>0$, $0<a\leqslant x\leqslant b<\infty$. Then (1.8) reduces to the Gamma operator

$$G_n(f, x)=\frac{x^{-n}}{(n-1)!}\int_0^{\infty} f\left(\frac{t}{n}\right) t^{n-1} e^{-t/x}\, dt. \tag{3.4}$$

Since $v(x)=x^2$, we get $M=b^2$. By Theorem 2,

$$\gamma=2\sum_{k=0}^{\infty}\left(1-\Phi\left(\frac{k}{b}\right)\right). \tag{3.5}$$

(4) *B*-SPLINE OPERATOR. Let Y_1 have density function

$$g(t, x)=\begin{cases}1 & t\in[x-\frac{1}{2}, x+\frac{1}{2}]\\ 0 & \text{otherwise.}\end{cases}$$

Then (1.8) defines the *B*-spline operator

$$L_n(f, x)=n\int_{x-1/2}^{x+1/2} f(t)\, B_n(t, x)\, dt, \tag{3.6}$$

where $-\infty<a\leqslant x\leqslant b<\infty$, $B_n(t, x)$ is the *B*-spline with knots $x-1/2$, $x+1/n-1/2, ..., x+1/2$ (see Dahmen and Micchelli [4]). Since $v(x)=1/12$, Theorem 2 gives

$$\gamma=2\sum_{k=0}^{\infty}(1-\Phi(2\sqrt{3}k))=1.000532. \tag{3.7}$$

(5) BASKAKOV OPERATOR. Let $P_x(Y_1=k)=pq^k$, $k=0, 1, 2, \ldots$ $(0 \leqslant p \leqslant 1,\ p+q=1)$. Put $p=(1+x)^{-1}$, then (1.8) defines the Baskakov operator

$$B_n^*(f, x)=(1+x)^{-n} \sum_{k=0}^{\infty} f\left(\frac{k}{n}\right)\binom{n+k-1}{k}\left(\frac{x}{1+x}\right)^k. \tag{3.8}$$

If $x \in [0, a]$, by $v(x)=x(1+x)$, $M=a(1+a)$. By Theorem 2,

$$\gamma=2 \sum_{k=0}^{\infty}\left(1-\Phi\left(\frac{k}{\sqrt{a(1+a)}}\right)\right). \tag{3.8}$$

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